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BOUNDARY REGULARITY FOR SOME NONLINEAR ELLIPTIC DEGENERATE EQUA--ETC(U)

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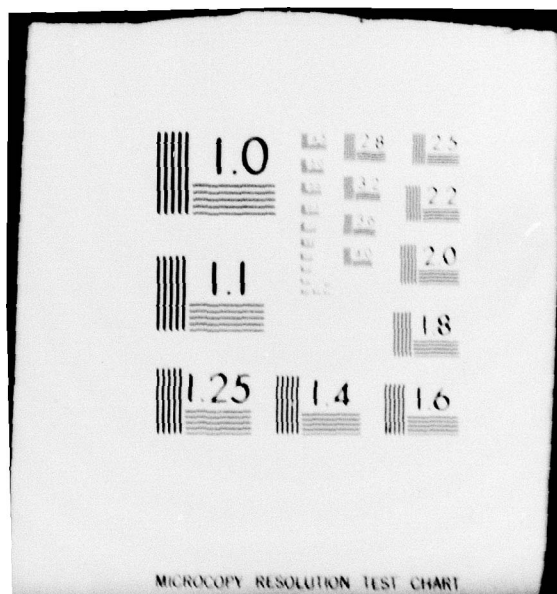
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*See 1473 in back.*

Haim Brezis and Pierre-Louis Lions

**LEVEL**

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BOUNDARY REGULARITY FOR SOME  
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Haim Brezis and Pierre-Louis Lions

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ABSTRACT

We consider the nonlinear elliptic degenerate equation

$$(1) \quad -x^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 2u = f(u) \quad \text{in } \Omega_a$$

where

$$\Omega_a = \{(x, y) \in \mathbb{R}^2, 0 < x < a, |y| < a\}$$

for some constant  $a > 0$  and  $f$  is a  $C^\infty$  function on  $\mathbb{R}$  such that  $f(0) = f'(0) = 0$ . Our main result asserts that: if  $u \in C(\bar{\Omega}_a)$  satisfies

$$(2) \quad u(0, y) = 0 \quad \text{for } |y| < a,$$

then  $x^{-2}u(x, y) \in C^\infty(\bar{\Omega}_{a/2})$  and in particular  $u \in C^\infty(\bar{\Omega}_{a/2})$ .

AMS (MOS) Subject Classifications: 35J60, 35J70

Key Words: Boundary regularity, nonlinear elliptic equations, degenerate elliptic equations.

Work Unit Number 1 - Applied Analysis

# SIGNIFICANCE AND EXPLANATION

Special solutions of the Yang-Mills field equations of theoretical physics may be obtained by solving a boundary value problem for a nonlinear elliptic equation in a two dimensional half space. This equation degenerates at the boundary of the region and this degeneracy makes it a delicate matter to study how the solutions behave near the boundary. In this work it is proved that the weak solutions previously known to exist are in fact smooth up to the boundary.

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BOUNDARY REGULARITY FOR SOME  
NONLINEAR ELLIPTIC DEGENERATE EQUATIONS

Haïm Brezis and Pierre-Louis Lions

1. Introduction

This paper deals with the question of boundary regularity of solutions of a nonlinear elliptic degenerate equation of the form

$$(1) \quad -x^2 \Delta u + 2u = f(u) \quad \text{in } \Omega_a$$

where

$$\Delta = D_x^2 + D_y^2$$

$$\Omega_a = \{(x, y) \in \mathbb{R}^2; \quad 0 < x < a, \quad |y| < a\}$$

for some constant  $a > 0$ , and  $f$  is a  $C^\infty$  function on  $\mathbb{R}$  such that

$$(2) \quad f(0) = f'(0) = 0 \quad .$$

Our main result is the following

Theorem 1. Assume  $u \in C^\infty(\bar{\Omega}_a) \cap C(\bar{\Omega}_a)$  satisfies (1) and

$$(3) \quad u(0, y) = 0 \quad \text{for } |y| < a \quad .$$

Then  $x^{-2}u(x, y) \in C^\infty(\bar{\Omega}_{a/2})$  and in particular  $u \in C^\infty(\bar{\Omega}_{a/2})$ .

Equation (1) occurs in the theory of multimeron solutions to Yang-Mills field equations (see [2]). More precisely the equation in [2] is:

$$-x^2 \Delta \psi + \psi^3 - \psi = 0 \quad \text{in } \Omega_a$$

together with the boundary conditions:

$$\psi(0, y) = \pm 1 \quad .$$

If we set  $u = \psi \mp 1$  we find

$$-x^2 \Delta u + (u \pm 1)^3 - (u \pm 1) = 0$$

that is (1) with  $f(u) = -u^3 + 3u^2$ . In [3] it is only proved that  $\psi$  is continuous up to the boundary (except at the points where  $\psi$  changes sign). Theorem 1 shows that  $\psi$  is  $C^\infty$  up to the boundary (except at the points where  $\psi$  changes sign).

We thank A. Jaffe for suggesting the problem and C. Goulaouic for useful discussions.

## 2. Some lemmas.

The proof relies on some lemmas

Lemma 2. Assume  $u \in C^2(\Omega_a) \cap C(\bar{\Omega}_a)$  satisfies:

$$(4) \quad |-x^2 \Delta u + 2u| \leq \alpha(u^2 + x^4) \quad \text{on } \Omega_a$$

for some constant  $\alpha$

$$(5) \quad u(0, y) = 0 \quad \text{for } |y| < a.$$

Then, there is a constant  $\beta$  such that

$$|u(x, y)| \leq \beta x^2 \quad \text{on } \Omega_{a/2}.$$

Proof of Lemma 2. For  $b < a$  set

$$M_b = \sup_{\Omega_b} |u|.$$

Since by (5)  $M_b \rightarrow 0$  as  $b \rightarrow 0$ , we may fix  $b$  so small that

$$(6) \quad \alpha b^2 < 1/2$$

$$(7) \quad \alpha M_b < 1/400.$$

We shall establish that

$$(8) \quad |u(x, 0)| \leq Ax^2 \quad \text{for } 0 < x < b,$$

where

$$(9) \quad A = \max\{\alpha b^2, \frac{100 M_b}{b^2}\}.$$

The conclusion of Lemma 2 follows easily. In order to prove (8) we introduce the function

$$(10) \quad v(x, y) = Ax^2 - Bx^4 + Cy^4$$

where  $A$  is defined by (9),

$$(11) \quad B = \frac{A}{2b^2} ,$$

$$(12) \quad C = \frac{M_b}{b^4} .$$

A direct computation shows that

$$(13) \quad -x^2 \Delta v + 2v \geq \alpha(v^2 + x^4) \text{ on } \Omega_b ,$$

$$(14) \quad v(x, \pm b) \geq M_b \text{ for } 0 < x < b ,$$

$$(15) \quad v(b, y) \geq M_b \text{ for } 0 < y < a ,$$

$$(16) \quad \alpha \sup_{\Omega_b} v \leq 1$$

$$(17) \quad v \geq 0 \text{ on } \Omega_b .$$

We now derive, using the maximum principle that

$$(18) \quad u \leq v \text{ on } \Omega_b .$$

Indeed by (14) and (15),  $u \leq v$  on  $\partial\Omega_b$ .

Suppose, by contradiction, that  $(u-v)$  achieves a positive maximum at  $(x_0, y_0) \in \Omega_b$ . We would have

$$\Delta(u-v)(x_0, y_0) \leq 0 .$$

On the other hand, we deduce from (4) and (13) that

$$-x^2 \Delta(u-v) + 2u - 2v \leq \alpha(u^2 - v^2) \text{ on } \Omega_b .$$

Therefore

$$2 \leq \alpha \{u(x_0, y_0) + v(x_0, y_0)\} .$$

$$\leq \alpha M_b + 1 \quad (\text{by (16)})$$

and thus  $\alpha M_b \geq 1$  — a contradiction with (7).

Lemma 3. Under the assumptions of Theorem 1 there exist constant  $\beta_k$  such that

$$|D_y^k u(x, y)| \leq \beta_k x^2 \quad \text{on } \Omega_{a/2} ,$$

for all  $k = 0, 1, 2, \dots$

Proof of Lemma 3. Since  $f(0) = 0$  we have

$$|f(u)| \leq C|u| \quad \text{on } \Omega_a$$

and by (1)

$$|\Delta u| \leq (C + 2) \frac{|u|}{x^2} \quad \text{on } \Omega_a .$$

It follows from Lemma 2 that  $\Delta u \in L^\infty(\Omega_{a/2})$ . We deduce from the  $L^p$  regularity theory (see e.g. [1]) that  $u \in C^1(\bar{\Omega}_{a/4})$ . In particular  $D_y u \in C(\bar{\Omega}_{a/4})$  and

$$D_y u(0, y) = 0 \quad \text{for } |y| < a/4$$

(since  $u(0, y) = 0$  for  $|y| < a$ ). Also, differentiating (1) with respect to  $y$  we find

$$-x^2 \Delta(D_y u) + 2(D_y u) = f'(u) D_y u \quad \text{on } \Omega_a .$$

By (2) we have

$$|f'(u)| \leq C|u|$$

and from Lemma 2 we see that

$$|f'(u)| \leq C\beta x^2, \quad \text{on } \Omega_{a/2} .$$

Consequently

$$|f'(u)D_y u|^2 \leq C\beta(|D_y u|^2 + x^4) ,$$

and Lemma 2 applied to  $D_y u$  shows that

$$|D_y u| \leq \beta_1 x^2 \text{ on } \Omega_{a/8} .$$

The conclusion of Lemma 3 for  $k = 1$  follows directly. When  $k \geq 2$  we proceed in a similar way, by induction, differentiating (1)  $k$  times with respect to  $y$ .

Lemma 4. Assume  $\varphi \in C^2([0, a[) \cap C([0, a])$  satisfies

$$-x^2 D_x^2 \varphi(x) + 2\varphi(x) = h(x), \quad 0 < x < a ,$$

where  $h \in L^\infty(0, a)$ .

Set  $\psi(x) = x^{-2}\varphi(x)$ , then

$$D_x \psi(x) = -x^{-4} \int_0^x h(t) dt, \quad 0 < x < a .$$

Proof. Indeed we necessarily have

$$\varphi(x) = \frac{C_1}{x} + C_2 x^2 + x^2 \int_x^a \frac{ds}{s^4} \int_0^s h(t) dt$$

for some constants  $C_1$  and  $C_2$ . Since the last term remains bounded as  $x \rightarrow 0$  we must take  $C_1 = 0$ , and the conclusion follows.

### 3. Proof of Theorem 1.

We have by (1)

$$-x^2 D_x^2 u + 2u = x^2 D_y^2 u + f(u) .$$

Let  $v(x, y) = x^{-2}u(x, y)$ . We deduce from Lemma 4 that

$$(19) \quad D_x v(x, y) = -x^{-4} \int_0^x [t^2 D_y^2 u(t, y) + f(u(t, y))] dt .$$

Set  $g(u) = u^{-2}f(u)$  so that by (2),  $g$  is a  $C^\infty$  function on  $\mathbb{R}$ . Changing the variable  $t$  in (19) into  $s = \frac{t}{x}$  we find

$$(20) \quad D_x v(x, y) = -x \int_0^1 [D_y^2 v(sx, y) + v^2(sx, y) g(s^2 x^2 v(sx, y))] s^4 ds.$$

It follows from Lemma 3 (applied with  $k = 0$  and  $k = 2$ ) that

$$(21) \quad |D_x v(x, y)| \leq C|x| \quad \text{on } \Omega_{a/2}.$$

Next, if we differentiate (2)  $k$  times with respect to  $y$  we obtain, using Lemma 3, that

$$(22) \quad |D_x D_y^k v(x, y)| \leq C_k \quad \text{on } \Omega_{a/2},$$

for all  $k$ .

We may now differentiate (20) once with respect to  $x$  and  $k$  times with respect to  $y$  and we find that

$$|D_{xx} D_y^k v(x, y)| \leq C_k \quad \text{on } \Omega_{a/2}$$

for all  $k$ . Proceeding by induction we obtain estimates for  $D_x^i D_y^k v$  and the conclusion of Theorem 1 follows (note that we have even an estimate of the form  $|D_x^i D_y^k v(x, y)| \leq Cx$  when  $i$  is odd).

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ABSTRACT (continued)

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